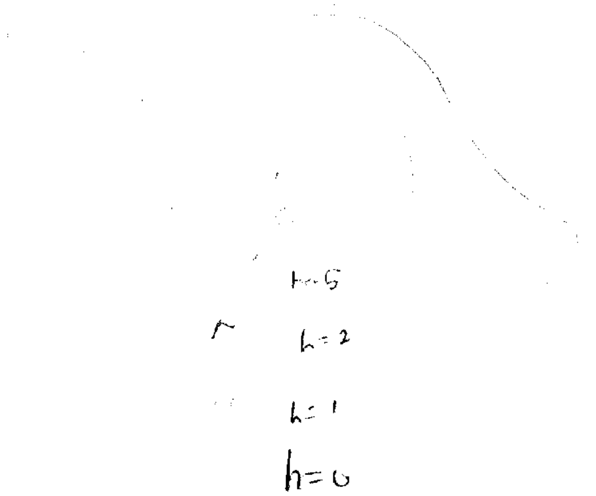
# Electric Potential

## I the gradient

Consider a top. map:




lines are flat surfaces — if you walk on them, your gravity doesn't change any … → equipotential lines in grav (mgh is the same)

Now suppose you want to get to the top. What is the fastest way? A: direction of greatest increase!

This will be normal to the lines at every point.

How to find this direction? Take derivative!

vector points in direction $\vec{G} = \frac{\partial h}{\partial x}\hat{i} + \frac{\partial h}{\partial y}\hat{j}$

$$\vec{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$ gradient in rect. coords (3D)

in spher. $\nabla t = \frac{\partial t}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial t}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial t}{\partial \phi}\hat{\phi}$

The gradient tells the direction of greatest increase of a scalar.

see Q — what does it mean if $\vec{\nabla} v = 0$?

notice $\vec{\nabla} v = \vec{v}$
scalar vector

stop

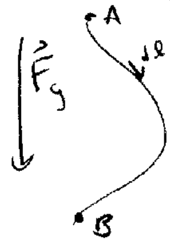
# Electric Potential

## II Electic Pot'l Energy

A. Work done by a conservative force:

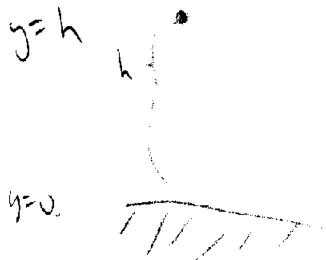
↳ path independent

eg gravity




work done by gravity → $\vec{W} = \int \vec{F} \cdot d\vec{\ell}$

gravity always points down, so dot product becomes




$W_g = \int_A^B F_g dh$ — gravity doesn't depend on height near earth (approx)

$W = mgh$

potential energy diff $= W = U_h - U_0 = mgh$

so $\Delta U = (U_0 - U_h) = -W = -mgh$

if $U_0 = 0$

$\Delta U = -U_h = -mgh$

$U_h = mgh$

note: when talking about potential energy, only the difference or change in energy is a meaningful quantity.

in general, $W = -\Delta U$ ← the change in potential energy associated w/ that force

↑ work done by a conservative force

B. Defn. The change in electric potential energy, when a charge moves from pt a to pt b is defined as the negative of the work done by the electric force to move the charge from a to b

$$\Delta U = U_b - U_a = -W_{ba} = -\int_a^b \vec{F} \cdot d\vec{l}$$

$$\Delta U = -\int_a^b \vec{F} \cdot d\vec{l}$$

recal; we defined quantity $\vec{E} = \frac{\vec{F}}{q}$

sub in:

$$\Delta U = -\int_a^b q\vec{E} \cdot d\vec{l}$$

similarly, it is useful to define a new quantity called electric potential difference:

$$\Delta V = \frac{\Delta U}{q} = \frac{-\int_a^b \cancel{q}\vec{E} \cdot d\vec{l}}{\cancel{q}} = -\int_a^b \vec{E} \cdot d\vec{l}$$

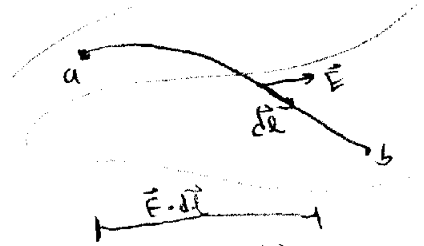
III

in your book

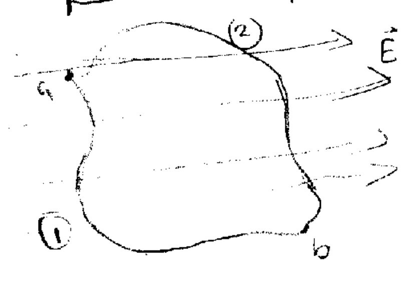
$$\boxed{\text{potential difference:} \quad V_{ba} = \Delta V = V_b - V_a = -\int_a^b \vec{E} \cdot d\vec{l}}$$

line integral

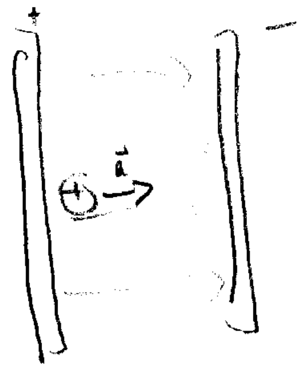
since force is conservative, work only depends on position of the path in the direction of the field - hence the dot product




consider:



example:



E field will do work - accel. charge. Kinetic energy comes from potential energy.

More charge has more pot'l energy, so it is useful to describe system by pot'l E/charge, V. Measured in Volts J/C

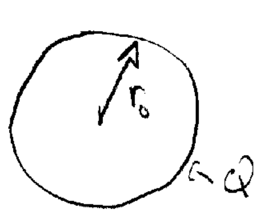
Some typical voltages:

| | |
|---|---|
| Thunder cloud | $10^8$ V |
| Power Line | $10^6$ V |
| Outlet | $10^2$ V |
| Flashlight | 1.5 V |

Batteries maintain a pot'l diff.

Example: (23-4) Charged conducting sphere

find V for (a) $r > r_0$, (b) $r = r_0$, (c) $r < r_0$




solution

(a) $r > r_0$ $\quad \vec{E} = \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2} \hat{r}$

$$V_b - V_a = -\int_a^b \vec{E} \cdot d\vec{l}$$

let $V=0$ at $r=\infty$, choosing $V_b=0$ for $r_b=\infty$

since $\vec{E}=E\hat{r}$, $\vec{E}\cdot d\vec{l} = E\hat{r}\cdot d\vec{l} = E\,dr$

pick out component of $d\vec{l}$ in $r$ direction

$$\overset{\nearrow 0}{V_b} - V_a = -\frac{1}{4\pi\epsilon_0} Q \int_r^{r_b=\infty} \frac{1}{r^2}\,dr$$

$$-V_a = -\frac{1}{4\pi\epsilon_0} Q \left[-\frac{1}{r}\right]_r^{\infty}$$

$$= -\frac{Q}{4\pi\epsilon_0}\left[0+\frac{1}{r}\right.$$

$$\boxed{V_{r>r_0} = V_a = \frac{Q}{4\pi\epsilon_0}\frac{1}{r}}$$

b) $r=r_0$ $\boxed{V_{r=r_0} = \frac{Q}{4\pi\epsilon_0}\frac{1}{r_0}}$

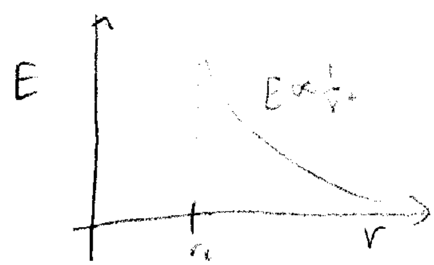
c) $r<r_0$ $Q_{encl}=0$ so $E=0$

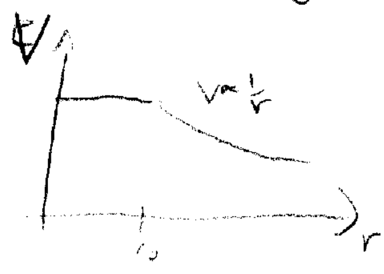
thus $V_b - V_a = V_{r=r_0} - V_r = -\cancel{\int E\,dl}^{\,0}$

$$V_{r=r_0} = V_r$$

so $\boxed{V_{r<r_0} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r_0}}$

notice that $E=0$ does not necessarily mean $V=0$



# IV Going the other way: finding $\vec{E}$ from $V$

vector ↑ ↑ scalar

$$V_b - V_a = -\int_a^b \vec{E}\cdot d\vec{l}$$

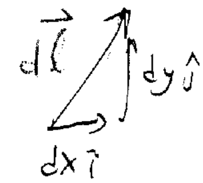
take away integral:

$$dV = -\vec{E}\cdot d\vec{l}$$

$$\vec{E} = E_x\hat{i} + E_y\hat{j} + E_z\hat{k}$$

$$d\vec{l} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$$

2D:




$$\vec{E}\cdot d\vec{l} = E_x dx + E_y dy + E_z dz$$

so $dV = -E_x dx + E_y dy + E_z dz$

the total derivative $dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz$

so we have (setting equal)

$$\frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz = -E_x dx - E_y dy - E_z dz$$

so $E_x = -\frac{\partial V}{\partial x}$ $\quad E_y = -\frac{\partial V}{\partial y}$ $\quad E_z = -\frac{\partial V}{\partial z}$

so $\vec{E} = -\left(\frac{\partial V}{\partial x}\hat{i} + \frac{\partial V}{\partial y}\hat{j} + \frac{\partial V}{\partial z}\hat{k}\right)$ look familiar?

$$\boxed{\vec{E} = -\vec{\nabla} V}$$

so electric field points in the direction of greatest decrease in potential

Example: E outside sphere of charge.

Found $V = \frac{Q}{4\pi\epsilon_0}\frac{1}{r}$

in spherical, $\vec{\nabla} = \frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial\theta}\hat{\theta} + \frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\hat{\phi}$

if we take $V_b = 0$ at $\infty$

$$0 - V_a = \frac{Q}{4\pi\epsilon_0}\left(\cancel{\frac{1}{\infty}}^{\,0} - \frac{1}{r_a}\right)$$

$$V_a = \frac{Q}{4\pi\epsilon_0 r_a}$$

so at any distance from a pt charge,

$$V = \frac{Q}{4\pi\epsilon_0 r}$$ "coulomb potential"

As you would expect this can be used to enable us to find the potential from many pts via the principle of superposition:

$$V_{tot} = \sum_i \frac{q_i}{4\pi\epsilon_0 r_i} = \frac{1}{4\pi\epsilon_0}\sum_i \frac{q_i}{r_i}$$

Electric Potential for any charge Distribution

$$\boxed{V = \frac{1}{4\pi\epsilon_0}\int \frac{dq}{r}}$$

$$= k\int \frac{dq}{r}$$

notice
- like coulomb's law, only $\frac{1}{r}$
- no vectors involved, since V is a scalar
- we assumed $V = 0$ at infinity as our point of reference

so $\vec{E} = -\vec{\nabla} V$

$$= -\frac{\partial V}{\partial r}\hat{r} + \cancelto{0}{\frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\theta}} + \cancelto{0}{\frac{1}{\sin\theta}\frac{\partial V}{\partial \phi}\hat{\phi}}$$

$$= -\frac{Q}{4\pi\epsilon_0}\frac{\partial}{\partial r}\frac{1}{r}\hat{r}$$

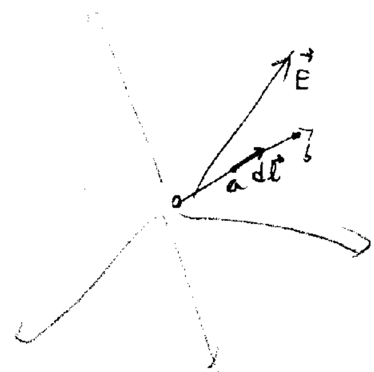
$$\boxed{\vec{E} = \frac{Q}{4\pi\epsilon_0}\frac{1}{r^2}\hat{r}}$$

section 23-3, 4 [illegible]

## Electric Pot'l Due to Pt charges (23-3)

For a pt charge, $E = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2} = k\frac{Q}{r^2}$

$$V_b - V_a = -\int_{r_a}^{r_b} \vec{E} \cdot d\hat{\ell}$$




$$d\vec{\ell} = d\vec{r}$$

$$\vec{E} \cdot d\vec{r} = E\,dr$$

$$V_b - V_a = -\int_{r_a}^{r_b} E\,dr$$

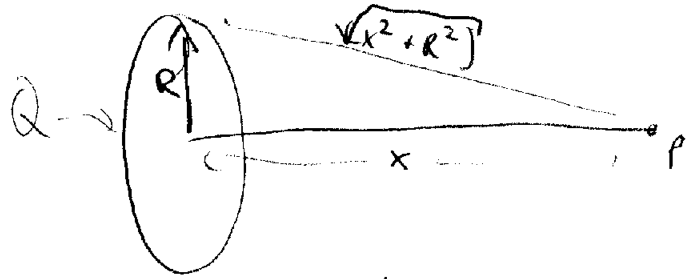
$$= \frac{-1}{4\pi\epsilon_0} Q \int_{r_a}^{r_b} \frac{1}{r^2}\,dr$$

$$= \frac{+1}{4\pi\epsilon_0} Q \left[+\frac{1}{r}\right]_{r_a}^{r_b}$$

$$= \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{r_b} - \frac{1}{r_a}\right)$$

Example:

ring of charge




$$V = k\int \frac{dq}{r}$$

$r = \sqrt{x^2+R^2}$ = constant, and not a vector!

$dq$ = who cares?

$$V = k\int \frac{dq}{\sqrt{x^2+R^2}}$$

$$= \frac{k}{\sqrt{x^2+R^2}}\int dq$$

$$= \frac{kQ}{\sqrt{x^2+R^2}}$$

in case someone does...

def $dq = \lambda\, d\ell$

$$\lambda = \frac{Q}{2\pi R}$$

$$V = \frac{k}{r}\frac{Q}{2\pi R}\int d\ell$$

$$= \frac{k}{r}\frac{Q}{\cancel{2\pi R}}\cancel{2\pi R}$$

$$= \frac{kQ}{r} \checkmark$$

# The Electron Volt + Electrostatic Potential E.

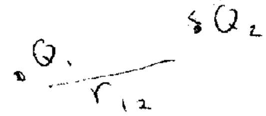
Creating a system/distn of charges takes work;
thus every system has static energy $U$

$$\Delta U = U_b - U_a = q(V_b - V_a)$$

Start with single pt charge $Q_1$; no pot'l energy
If 2nd charge brought in, the potential due to $Q_1$

is $V = k\dfrac{Q_1}{r_{12}}$ (and $V=0$ at $\infty$ so)

$$U = Q_2 V = k\frac{Q_1 Q_2}{r_{12}}$$




Bring in 3rd charge:
requires energy $k\dfrac{Q_1 Q_3}{r_{13}} + k\dfrac{Q_2 Q_3}{r_{23}}$

So system has PE

$$U = k\left(\frac{Q_1 Q_2}{r_{12}} + \frac{Q_1 Q_3}{r_{13}} + \frac{Q_2 Q_3}{r_{23}}\right) \quad (U=0 \text{ at } r=\infty)$$

etc.

New unit better size:
take particle with charge $e$

move it across pot'l dif of 1V
It acquires 1eV of energy:

$$\boxed{1\,eV = qV = (1.6\times10^{-19}\,C)(1.0\,V) = 1.6\times10^{-19}\,J}$$